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HÖLDER ESTIMATES AND REGULARITY FOR HOLOMORPHIC AND HARMONIC FUNCTIONS

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Abstract

In this paper, we proved that if a singular manifold satisfies a weak mean value property for positive subharmonic functions then one can derive an oscillation bound for bounded holomorphic functions. Moreover, if we further assume that the volume decays at most polynomially at a singular point, then we obtain a Hölder estimate of the holomorphic function at that point. In a similar spirit, we also established a continuity estimate for bounded harmonic functions, with a finite dimensional exception, at a singular point of a manifold satisfying a weak mean value property and a polynolmial volume decay condition.

0. Introduction

The theory of DeGiorgi-Nash-Moser asserts that if $f \geq 0$ is a non-constant solution to the partial differential equation

$$Lf = 0$$

on a ball $B(2R) \subset \mathbb{R}^n$, with

$$L = \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right)$$

being a uniformly elliptic operator of divergence form, then f must satisfy the Harnack inequality

$$\sup_{B(R)} f \leq C \inf_{B(R)} f$$

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for some constant C depending only on the ellipticity bounds of (a_{ij}) and the dimension n. An upshot of the Harnack inequality is the oscillation estimate, which states that there exist constants $C > 0$ and $0 < \alpha \leq 1$, such that if

$$\omega(r) = \sup_{B(r)} f - \inf_{B(r)} f$$

denotes the oscillation of f over $B(r)$, then

$$\omega(r) \leq C \left(\frac{r}{R}\right)^{\alpha} \omega(R). \tag{0.1}$$

In particular, by letting $r \to 0$, this implies the C^{α} regularity of f.

On the other hand, if f is defined on the whole $\mathbb{R}^n$, by letting $R \to \infty$ while keeping r fixed, this implies f must be constant if f satisfies the growth condition

$$|f(x)| = o(|x|^{\alpha}).$$

For simplicity, let us refer this property as the *α-Liouville property.* One can think of the α-Liouville property as the dual of the C^{α} regularity, or perhaps C^{α} regularity at ∞.

Taking this on a manifold setting, by unraveling Moser's original argument [16, 4], one notes that the manifold only needs to satisfy a Poincaré inequality, a Sobolev inequality, and volume doubling property in order for Moser's program to go through. The essence was further illuminated by the works of Saloff-Coste [14, 15] and Grigor'yan [3], that a Poincaré inequality and a volume doubling property suffice to imply the Harnack inequality, hence the α-Liouville property and the C^{α} regularity.

In [5], the first author proved if a manifold satisfies a mean value inequality and a volume comparison condition, then the space of harmonic functions that grow at most polynomially of degree d must be finite dimensional. Later in [7], the authors gave an even weaker hypothesis to ensure the finite dimensionality for harmonic functions. In fact, the arguments in [5] and [7] are also valid for solutions of uniformly elliptic operators of both divergence and non-divergence form. The authors showed that if a manifold M satisfies a weak mean value property (Definition 1) and has polynomial volume growth, then the space of harmonic functions on M satisfying the growth condition

$$|f(x)| = O(r^d(x))$$

for some $d > 0$ must be finite dimensional. Consequently, as observed in [7], if the manifold is in addition Kähler, then there are no non-constant bounded holomorphic functions defined on M. This hinges on the simple fact that powers of a holomorphic function are also holomorphic. Note that since the Sobolev inequality alone implies both the weak mean value property by Moser's iteration and the polynomial volume growth [7], one concludes that the space of polynomial growth harmonic functions of fixed degree must be finite dimensional on any complete manifold satisfying Sobolev inequality. They also addressed the issues of sharpness of the dimension estimate in their subsequent papers [8] and [9].

On the other hand, if C^α regularity is dual to the α-Liouville property, then one may ask if any local regularity property can be asserted from the finite dimensionality of the space of polynomial growth harmonic functions of fixed degree. A simple case to examine is the holomorphic function case where we have the convenience of taking powers. The purpose of this paper is to prove (Theorem 4) that if a complete Kähler manifold satisfies the weak mean value property then an estimate of the type (0.1) holds with the constant C depending on the ratio of the volumes of balls. Moreover, if at a point the manifold has at most polynomial volume decay for small balls, then it will have C^α regularity (Corollary 5). On the other hand, if the manifold has at most polynomial volume growth for large balls, then it has the α- Liouville property (Corollary 8) for holomorphic functions. In fact, we will obtain a uniform C^α estimate for all holomorphic functions (Corollary 5) if we assume a volume comparison condition.

A typical situation to apply Theorem 4 is when the Kähler manifold is a singular algebraic or minimal variety. In this case, the subvariety inherits a Sobolev inequality from the ambient manifold [11] hence possesses the weak mean value property. However, in order to deal with this situation, the balls in Theorem 4 are usually taken to be extrinsic balls rather than geodesic balls. All the argument in Theorem 4 will carry through with the one exception that extrinsic balls are not necessarily connected. In order to overcome this, the assumption of local irreducibility (Definition 6) is imposed at the point in question. In fact, the local irreducibility assumption is necessary as can be seen by taking M as the union of the z and w planes in $\mathbb{C}^2$. The singular point $p = (0, 0)$ disconnects the two copies of $\mathbb{C}$ hence violates the local irreducibility assumption. Moreover, the function given by 1 on the z-plane and 0 on the w-plane will be bounded holomorphic function on M that is discontinuous.

Also note that since any complete manifold with Ricci curvature bounded from below possesses the weak mean value property locally [6], one can also interpret the local C^α estimate as a weaker form of Schwarz lemma (see [17], [2], and [13]) on these singular spaces.

In the case of harmonic functions, since it is possible to have a finite dimensional space of nonconstant bounded harmonic functions on a complete manifold with the weak mean value property and polynomial volume growth, we do not expect to have a C^α regularity theory which holds for all harmonic functions. However, we do expect that for each singular point p of M, other than a finite dimensional exception, all harmonic functions are C^α. While this is the ideal local analog for the global theory, we only managed to prove this theorem for C^0 regularity in Theorem 9. We would like to point out that the local irreducibility assumption is not necessary in Theorem 9. The finite dimension exception allows some disconnect components for the complement of the singular set of M. Moreover, the mean value constant λ also puts a restriction on the number of disconnected components.

In the last section, we proved a finite dimensionality theorem on the space

$$\mathcal{H}_d(L, M) = \left\{ f \mid Lf = 0 \text{ and } |f|(x) = O(r^d(x)) \right\}$$

of L-harmonic functions with at most polynomial growth of degree d defined on a manifold with nonnegative Ricci curvature. The operator L is an elliptic operator of divergence form with

$$Lf = \operatorname{div}(A(x)\nabla f)$$

where $A : TM \to TM$ is an endomorphism of the tangent bundle. The assumption on $A(x)$ is given by measurable coefficients in local coordinates and also

$$0 < \phi(r)\, I \leq A(x) \leq \Phi(r)\, I < \infty$$

for all $x \in B_p(\frac{3r}{2}) \setminus B_p(\frac{r}{2})$ with

$$\frac{\Phi(r)}{\phi(r)} \leq \Omega$$

for some fixed constant $\Omega > 0$. Note that this assumption on A allows the operator to grow or decay polynomially as long as the lower bound and the upper bound are of the same order. Examples of these is when

$$A(x) = r^t(x)\, A_0$$

for some $-\infty < t < \infty$.

We would like to point out that when A is uniformly bounded on the entire space and further assuming that the operator has an asymptotic limit, the issue of finding the dimension of $\mathcal{H}_d(L, \mathbb{R}^n)$ was studied by Avellaneda-Lin [1], Moser-Struwe [12], and Lin [10]. For general uniformly elliptic operators of divergence and non-divergence form, the issue of finite dimensionality of $\mathcal{H}_d(L, M)$ and to estimate its dimension was studied by the first author in [5] and then by the authors in [7] and [9]. Theorem 10 is the first situation which the coefficients are not required to be uniformly bounded over the whole manifold.

1. Preliminaries

Let us first recall the weak mean value property.

Definition 1. A Riemannian manifold M is said to satisfy the *weak mean value property* WM(R) if there exist constants $b \geq 1$ and $\lambda \geq 1$ such that

$$u^2(q) \leq \frac{\lambda}{V_q(r)} \int_{B_q(br)} u^2$$

for all $r \leq R$ and subharmonic function u defined on the geodesic ball $B_q(bR)$, where $V_q(r)$ denotes the volume of the ball $B_q(r)$.

As pointed out before, a complete manifold with Ricci curvature bounded from below possesses the weak mean value property WM(R) for any fixed R [6]. More generally, Moser's iteration implies that a complete manifold M possesses the weak mean value property WM(R) if the following type of Sobolev inequality $S(R, \nu)$ is valid on M:

$$\left(\int_{B_p(r)} |f|^{\frac{2\nu}{\nu-2}} \right)^{\frac{\nu-2}{\nu}} \leq B\, V_p^{-\frac{2}{\nu}}(r) \int_{B_p(r)} (r^2\, |\nabla f|^2 + f^2)$$

for all $p \in M$, $0 < r \leq R$ and for all $f \in H_0^1(B_p(r))$, where $B > 0$ and $\nu > 2$ are constants.

The following lemma is a simplified version of a lemma used in [5]. It is a key ingredient in most of the finite dimensionality proofs.

Lemma 2. *Let M be a Riemannian* (*not necessarily complete*) *manifold satisfying the weak mean value property* WM(R)*. Let K be a k-dimensional linear space of complex functions on M with the property that*

$$\Delta |u|^2 \geq 0$$

for all $u \in K$. If we denote by

$$A_t(u,v) = \int_{B_p(t)} u\bar{v}$$

the Hermitian inner product on K, then the trace of A_r with respect to the inner product $A_{(2b+1)r}$ must satisfy

$$\mathrm{tr}_{A_{(2b+1)r}} A_r \leq \lambda$$

for all $r \leq \frac{R}{2}$.

Proof. Let $\{f_1, \ldots, f_k\}$ be a unitary basis with respect to the Hermitian inner product $A_{(2b+1)r}$. For any $\epsilon > 0$ sufficiently small, let $q \in B_p(r)$ be a point such that

$$(1+\epsilon) \sum_{i=1}^{k} |f_i|^2(q) \geq \sup_{B_p(r)} \sum_{i=1}^{k} |f_i|^2. \tag{1.1}$$

By a unitary transformation if necessary, we may assume that $f_i(q) = 0$ for all $i \geq 2$ and

$$\sum_{i=1}^{k} |f_i|^2(q) = |f_1|^2(q).$$

Applying the weak mean value property, we have

$$\begin{aligned} \mathrm{tr}_{A_{(2b+1)r}} A_r &= \int_{B_p(r)} \sum_{i=1}^{k} |f_i|^2 \\ &\leq V_p(r) \sup_{B_p(r)} \sum_{i=1}^{k} |f_i|^2 \\ &\leq V_p(r)\,(1+\epsilon)\,|f_1|^2(q) \\ &\leq \lambda\,(1+\epsilon)\,V_p(r)\,V_q(2r)^{-1} \int_{B_q(2br)} |f_1|^2 \\ &\leq \lambda\,(1+\epsilon) \int_{B_p((2b+1)r)} |f_1|^2 \\ &= \lambda\,(1+\epsilon). \end{aligned} \tag{1.2}$$

The lemma follows by letting $\epsilon \to 0$. q.e.d.

Let us remark that when M is a complete, smooth Riemannian manifold, then the maximum principle asserts that there exists $q \in \partial B_p(r)$ such that

$$\sum_{i=1}^{k} |f_i|^2(q) = \sup_{B_p(r)} \sum_{i=1}^{k} |f_i|^2. \tag{1.3}$$

In this case, we can replace (1.1) by (1.3). Moreover, if we replace (1.2) by

$$\begin{aligned}
\mathrm{tr}_{A_{\frac{3r}{2}}} A_r &= \int_{B_p(r)} \sum_{i=1}^{k} |f_i|^2 \\
&\leq V_p(r) \sup_{B_p(r)} \sum_{i=1}^{k} |f_i|^2 \\
&= V_p(r)\, |f_1|^2(q) \\
&\leq \lambda\, V_p(r)\, V_q(\frac{r}{2b})^{-1} \int_{B_q(\frac{r}{2})} |f_1|^2 \\
&\leq \lambda\, V_p(r)\, V_q(\frac{r}{2b})^{-1} \int_{B_p(\frac{3r}{2})} |f_1|^2,
\end{aligned}$$

we conclude that

$$\mathrm{tr}_{A_{\frac{3r}{2}}} A_r \leq \frac{\lambda\, V_p(r)}{V_q(\frac{r}{2b})} \tag{1.4}$$

where we only need to assume that the weak mean value inequality

$$u^2(q) \leq \frac{\lambda}{V_q(\frac{r}{2b})} \int_{B_q(\frac{r}{2})} u^2 \tag{1.5}$$

is valid for points q which is distance r from p. This observation allows us to consider elliptic operators of divergence form that may not have uniformly bounded coefficients. We will address this issue in §5.

Lemma 3. *Let M be a Riemannian (not necessarily complete) manifold satisfying the weak mean value property* WM(R). *Let K be a k-dimensional linear space of complex functions on M with the property that*

$$\Delta |u|^2 \geq 0$$

for all $u \in K$. If $k > \lambda$ and if

$$A_t(u,v) = \int_{B_p(t)} u\bar{v}$$

denotes the Hermitian inner product on K, then the determinant of A_r with respect to the inner product A_R must satisfy

$$\det{}_{A_R} A_r \leq \left(\frac{k\, r^\beta}{\lambda\, R^\beta}\right)^k$$

for all $r \leq R$, where

$$\beta = \frac{\ln k - \ln \lambda}{\ln(2b+1)}.$$

Proof. Lemma 2 and the arithmetic-geometric mean inequality assert that

$$\begin{aligned}\det{}_{A_{(2b+1)r}} A_r &\leq \left(\frac{\mathrm{tr}_{A_{(2b+1)r}} A_r}{k}\right)^k \\ &\leq \left(\frac{\lambda}{k}\right)^k.\end{aligned}$$

Setting $\gamma = (2b+1)^{-1}$ and $r = \gamma^i R$ for $i = 0, 1, \ldots,$ we can rewrite this as

$$\det{}_{A_{\gamma^i R}} A_{\gamma^{i+1} R} \leq \left(\frac{\lambda}{k}\right)^k.$$

Iterating this inequality j times by letting $i = 0, \ldots, j-1$, we have

$$\det{}_{A_R} A_{\gamma^j R} \leq \left(\frac{\lambda}{k}\right)^{jk}.$$

In particular, we have

$$\begin{aligned}\det{}_{A_R} A_r &\leq \left(\frac{r}{R}\right)^{\frac{k(\ln\lambda - \ln k)}{\ln\gamma}} \\ &= \left(\frac{r}{R}\right)^{\frac{k(\ln k - \ln\lambda)}{\ln(2b+1)}}\end{aligned}$$

if $r = \gamma^j R$ for some $j = 1, 2, \ldots.$ When $\gamma^{j+1} R < r < \gamma^j R$, we observe that

$$\det{}_{A_R} A_r = \det{}_{A_R} A_{\gamma^j R} \det{}_{A_{\gamma^j R}} A_r.$$

By choosing an unitary basis $\{f_1, \ldots, f_k\}$ with respect to $A_{\gamma^j R}$ that diagonalizes A_r, we note that

$$\begin{aligned}\det_{A_{\gamma^j R}} A_r &= \prod_{i=1}^{k} \int_{B_p(r)} |f_i|^2 \\ &\leq \prod_{i=1}^{k} \int_{B_p(\gamma^j R)} |f_i|^2 \\ &= 1.\end{aligned}$$

Hence

$$\begin{aligned}\det_{A_R} A_r &\leq \det_{A_R} A_{\gamma^j R} \\ &\leq \left(\frac{\lambda}{k}\right)^{jk}.\end{aligned}$$

Using the fact that $\gamma^{j+1}R < r$ and assuming that $k > \lambda$, we get

$$\left(\frac{\lambda}{k}\right)^{jk} \leq \left(\frac{k}{\lambda}\right)^{k} \left(\frac{r}{R}\right)^{\frac{k(\ln k - \ln \lambda)}{\ln(2b+1)}},$$

which implies that

$$\det_{A_R} A_r \leq \left(\frac{k\, r^{\frac{(\ln k - \ln \lambda)}{\ln(2b+1)}}}{\lambda\, R^{\frac{(\ln k - \ln \lambda)}{\ln(2b+1)}}}\right)^{k}$$

for all $r \leq R$. q.e.d.

2. Holomorphic Functions

We are now ready to prove the Hölder estimate for holomorphic functions.

Theorem 4. *Let M be a Kähler (not necessarily complete) manifold satisfying the weak mean value property* WM(R)*. Suppose f is a holomorphic function defined on the geodesic ball $B_p(R)$. For all $r \leq R$ and for all $k+1 > \lambda$, if we denote $\omega(r)$ to be the oscillation of f on $B_p(r)$, then*

$$\omega(r) \leq Ck \left(\frac{V_p(R)}{V_p(r)} \left(\frac{r}{R}\right)^{\beta}\right)^{\frac{1}{k(k+1)}} \omega(R),$$

where $C > 0$ is a constant depending on λ and b alone, and

$$\beta = \frac{\ln(k+1) - \ln\lambda}{\ln(2b+1)}.$$

Proof. We may assume that the function f is nonconstant and normalized to have $\omega(R) = 1$, $\inf_{B_p(R)} |f| = 0$, and $\sup_{B_p(R)} |f| \leq 1$. Let us consider the linear space K spanned by the functions $\{1, f, f^2, \dots, f^k\}$. Then K is of complex dimension $k+1$. For $h \in K$ with $A_R(h,h) = 1$, there exist complex numbers $c_0, c_1, \dots, c_k$ such that $h = \sum_{i=0}^k c_i f^i$. Hence

$$\begin{aligned} 1 &= A_R(h,h) \\ &= \int_{B_p(R)} \left(\sum_{i=0}^k c_i f^i\right) \overline{\left(\sum_{i=0}^k c_i f^i\right)} \\ &\leq \int_{B_p(R)} \left(\sum_{i=0}^k |c_i|\right)^2 \\ &\leq V_p(R) \left(\sum_{i=0}^k |c_i|\right)^2, \end{aligned}$$

and

$$\left(\sum_{i=0}^k |c_i|\right)^2 \geq \frac{1}{V_p(R)}. \tag{2.1}$$

On the other hand, we can choose points z_0, $z_k \in B_p(r)$ such that

$$|f(z_0) - f(z_k)| \geq \frac{\omega(r)}{2}.$$

Since $f(B_p(r))$ is connected in the complex plane $\mathbb{C}$, for any $s \leq |f(z_0) - f(z_k)|$, we have

$$\partial B_{f(z_0)}(s) \cap f(B_p(r)) \neq \emptyset.$$

Therefore, for $i = 1, \dots, k-1$, we can choose point $z_i \in B_p(r)$ such that

$$f(z_i) \in \partial B_{f(z_0)}\left(\frac{i}{k}|f(z_0) - f(z_k)|\right).$$

In particular, we have

$$|f(z_i) - f(z_j)| \geq \frac{\omega(r)}{2k} \tag{2.2}$$

for any $i \neq j$ with $i, j = 0, 1, \ldots, k$.

Now consider the linear system

$$\sum_{i=0}^{k} c_i f^i(z_j) = h(z_j)$$

for $j = 0, \ldots, k$. Let A denote the matrix

$$A = \begin{pmatrix} 1 & f(z_0) & \ldots & f^k(z_0) \\ 1 & f(z_1) & \ldots & f^k(z_1) \\ & & \ddots & \\ 1 & f(z_k) & \ldots & f^k(z_k) \end{pmatrix}.$$

Then the linear system can be written as

$$A \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} h(z_0) \\ h(z_1) \\ \vdots \\ h(z_k) \end{pmatrix},$$

and

$$\begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_k \end{pmatrix} = A^{-1} \begin{pmatrix} h(z_0) \\ h(z_1) \\ \vdots \\ h(z_k) \end{pmatrix}.$$

Cramer's rule asserts that the $(ij)^{th}$ entry of A^{-1} is given by

$$\left|(A^{-1})_{ij}\right| = \left|\frac{\det M_{ij}}{\det A}\right|,$$

where M_{ij} is the $(ij)^{th}$ minor of A. Since $|f| \leq 1$, we have

$$|\det M_{ij}| \leq k!.$$

Also, by (2.2),

$$\begin{aligned} |\det A| &= \prod_{0 \leq i < j \leq k} |f(z_i) - f(z_j)| \\ &\geq \left(\frac{\omega(r)}{2k}\right)^{\frac{k(k+1)}{2}}. \end{aligned}$$

If we let $g(r) = \sup_{B_p(r)} |h|$, then

$$|c_i| \leq \frac{(k+1)\, k!\, (2k)^{\frac{k(k+1)}{2}}}{\omega(r)^{\frac{k(k+1)}{2}}} g(r).$$

Hence we conclude that

$$\sum_{i=0}^{k} |c_i| \leq \frac{(k+1)^2\, k!\, (2k)^{\frac{k(k+1)}{2}}}{\omega(r)^{\frac{k(k+1)}{2}}} g(r).$$

Combining with (2.1), we obtain the estimate

$$\frac{1}{V_p(R)} \leq \frac{(k+1)^4\, (k!)^2\, (2k)^{k(k+1)}}{\omega(r)^{k(k+1)}} g^2(r),$$

hence

$$g^2(r) \geq V_p(R)^{-1}\, (k+1)^{-4}\, (k!)^{-2}\, (2k)^{-k(k+1)}\, \omega(r)^{k(k+1)}. \tag{2.3}$$

On the other hand,

$$g^2(r) \leq 2|h(q)|^2$$

for some $q \in B_p(r)$ and by the weak mean value property, we have

$$\begin{aligned} g^2(r) &\leq 2|h(q)|^2 \\ &\leq \frac{2\lambda}{V_q(2r)} \int_{B_q(2br)} |h|^2 \\ &\leq \frac{2\lambda}{V_q(2r)} \int_{B_p((2b+1)r)} |h|^2. \end{aligned}$$

Thus, applying to (2.3), we conclude that

(2.4)

$$\begin{aligned} \int_{B_p((2b+1)r)} |h|^2 &\geq (2\lambda)^{-1}\, V_q(2r)\, g^2(r) \\ &\geq (2\lambda)^{-1}\, (k+1)^{-4}\, (k!)^{-2}\, (2k)^{-k(k+1)}\, \omega(r)^{k(k+1)}\, \frac{V_p(r)}{V_p(R)}. \end{aligned}$$

Since $h \in K$ with $A_R(h,h) = 1$ is arbitrary, (2.4) implies that

(2.5)

$$\begin{aligned} \det{}_{A_R} A_{(2b+1)r} \geq (2\lambda)^{-(k+1)}\, (k+1)^{-4(k+1)}\, (k!)^{-2(k+1)}\, (2k)^{-k(k+1)^2} \\ \cdot\, \omega(r)^{k(k+1)^2} \left(\frac{V_p(r)}{V_p(R)}\right)^{k+1}. \end{aligned}$$

On the other hand, Lemma 3 asserts that

$$\det_{A_R} A_{(2b+1)r} \leq \left(\frac{(k+1)\,((2b+1)r)^{\beta}}{\lambda\, R^{\beta}} \right)^{k+1}.$$

Hence combining with (2.5), we have

$$\begin{aligned}\omega(r) &\leq \left(2^{k^2+k+1}\,(k+1)^5\,(k!)^2\,k^{k(k+1)}\,(2b+1)^{\beta}\,\left(\frac{r}{R}\right)^{\beta}\,\frac{V_p(R)}{V_p(r)} \right)^{\frac{1}{k(k+1)}} \\ &\leq C\,k\,\left(\frac{V_p(R)}{V_p(r)}\,\left(\frac{r}{R}\right)^{\beta} \right)^{\frac{1}{k(k+1)}},\end{aligned}$$

where

$$\beta = \frac{\ln(k+1) - \ln\lambda}{\ln(1+2b)}.$$

This proves the theorem. q.e.d.

3. Applications

We will now apply the estimate of Theorem 4 to obtain Hölder estimate and α-Liouville property for holomorphic functions.

Corollary 5. *Let M be a Kähler (not necessarily complete) manifold satisfying the weak mean value property* WM(R)*. Suppose there exist constants $C_0(p,R) > 0$ and $\nu > 0$ such that*

$$C_0(p,R)\,V_p(R) \leq \left(\frac{R}{r}\right)^{\nu}\,V_p(r)$$

for all $r < R$. Then there exists $\alpha = \alpha(\nu,\lambda,b)$ depending only on the quantities ν, λ, and b such that if f is a holomorphic function defined on $B_p(R)$ then its oscillation $\omega(r)$ must satisfy

$$\omega(r) \leq c(\lambda, b, C_0(p,R), \nu)\,\omega(R)\,\left(\frac{r}{R}\right)^{\alpha}.$$

In particular, if the constant C_0 is independent of p and R, that is, there exist constants $C_1 > 0$ and $\nu > 0$ such that

$$C_1\,V_p(R) \leq \left(\frac{R}{r}\right)^{\nu}\,V_p(r)$$

for all $p \in M$ and $r < R$, then we obtain the uniform estimate

$$\omega(r) \leq c(\lambda, b, C_1, \nu)\,\left(\frac{r}{R}\right)^{\alpha}\,\omega(R).$$

Proof. Applying the volume lower bound to Theorem 4, we have

$$\omega(r) \leq Ck \left(\frac{r^{\beta-\nu}}{C_0\, R^{\beta-\nu}}\right)^{\frac{1}{k(k+1)}} \omega(R). \tag{3.1}$$

By choosing k sufficiently large such that $\beta \geq 2\nu$ this implies that

$$\omega(r) \leq c(\lambda, b, C_0, \nu) \left(\frac{r}{R}\right)^{\alpha} \omega(R).$$

This proves the first part of the corollary. The second part of the corollary follows immediately. q.e.d.

Observe that the above argument can be applied to the situation when M is a submanifold of another Riemannian manifold N. In this case, one will replace the intrinsic distance function by the extrinsic distance function. Moreover, the geodesic balls should be replaced by extrinsic balls obtained by taking geodesic balls of N intersecting M. Particular cases which the theorem may apply are complex subvarieties of a Kähler manifold or minimal submanifolds which are Kähler. In these cases, the Kähler variety M may be singular with the singular set given by Σ. In this situation, the theorem will apply to the incomplete manifold $M \setminus \Sigma$. Due to the fact that we are considering the extrinsic ball, the set $B_p(R) \setminus \Sigma$ may not be connected, hence a connectedness assumption is required.

Definition 6. A singular submanifold M with singular set Σ is said to be locally irreducible if for any extrinsic ball $B_p(r)$ with sufficiently small radius r, the set $B_p(r) \setminus \Sigma$ is connected.

Corollary 7. *Let M be a locally irreducible Kähler subvariety of a complete manifold N. Suppose M satisfies the weak mean value property* WM(R) *and the volume of extrinsic balls satisfy*

$$C_0\, r^{\alpha} \leq V_p(r)$$

for some constants $C_0 > 0$ and $\alpha > 0$, and for all $r \leq R$. Suppose f is a bounded holomorphic function defined on the extrinsic ball $B_p(R)$ with $p \in \Sigma$, then f must be Hölder continuous at the point p.

Note that when M is an algebraic variety, it is known that bounded holomorphic functions can be extended across the singular set if M is irreducible and locally irreducible. In this case, the variety inherits a Sobolev inequality from $\mathbb{CP}^N$, hence satisfies the weak mean value inequality. Moreover, the volume comparison condition is also satisfied with the constant involving the multiplicity of the variety. Hence Corollary 7 can be viewed as a generalization of this theorem.

Corollary 8. *Let M be a Kähler manifold satisfying weak mean value property* WM(R) *for all $R > 0$. Suppose that M has polynomial volume growth of order ν. Then there exists a positive number α such that every holomorphic function on M of growth order at most α must be constant.*

Proof. Since M has polynomial volume growth of order ν, we have

$$V_p(R) \leq C_2\, V_p(1)\, R^{\nu}$$

for all $R \geq 1$. Let f be a holomorphic function on M and $\omega(r)$ its oscillation on the ball $B_p(r)$. Then by Theorem 4, we conclude that there exists $\alpha > 0$ such that

$$\omega(1) \leq c\, R^{-\alpha}\, \omega(R)$$

for all $R \geq 1$. In particular, if f is of growth order strictly less than α, then by letting R go to infinity we get $\omega(1) = 0$. Thus, f must be constant by the unique continuation property of the holomorphic functions. q.e.d.

4. Continuity of Harmonic Functions

Although we are still unsuccessful in proving Hölder continuity for the real case — harmonic functions — we manage to prove that for any given point, other than a finite dimensional exception, all harmonic functions are continuous.

Theorem 9. *Let M be a singular Riemannian manifold satisfying the weak mean value property* WM(R)*. Let us denote the singular set of M by Σ. Let $p \in \Sigma$ be a fixed point with the property that the volume of the ball of radius r centered at p satisfies*

$$V_p(r) \geq C\, r^{\nu}$$

for some constants $C, \nu > 0$. Then there exists a k-dimensional subspace H of bounded harmonic functions defined on $B_p(1)$ with

$$k \leq \lambda\, (2b+1)^{\nu},$$

such that if f is a bounded harmonic function defined on the geodesic ball $B_p(1)$, then $f - \phi$ is continuous at p for some $\phi \in H$.

Proof. To prove the theorem, it suffices to show that for any subspace K of bounded harmonic functions defined on $B_p(1)$, with dimension $k > \lambda\,(2b+1)^\nu$, there must have at least one nonzero function $f \in K$ such that f is continuous at p. Arguing by contradiction, if this is not the case, then for each $f \in K$, there exists $\epsilon_f > 0$, such that the oscillation $\omega_f(r)$ of f on the ball $B_p(r)$ must satisfy

$$\omega_f(r) \geq \epsilon_f$$

for all r. Observe that since $\omega_f(r)$ is a monotonic non-increasing function of r, ϵ_f can be taken to be

$$\epsilon_f = \inf_{r\to 0} \omega_f(r).$$

We now claim that $\epsilon_f > 0$ is a continuous function defined on the $L^2(B_p(1))$-unit sphere $\mathbb{S}(K)$ of K. Then the fact that K is finite dimensional implies the set $\mathbb{S}(K)$ is compact and we can choose

$$\epsilon = \inf_{f\in\mathbb{S}(K)} \epsilon_f > 0$$

such that

$$\omega_f(r) \geq \epsilon$$

for all $r \leq 1$ and for all $f \in \mathbb{S}(K)$.

To prove the continuity of ϵ_f, we consider the function

$$f_t = f\,\cos t + h\,\sin t$$

where f and h are orthogonal in $\mathbb{S}(K)$. Obviously $f_t \in \mathbb{S}(K)$ and $f_t \to f$ as $t \to 0$. Then for $x, y \in B_p(r)$ and $0 \leq t \leq \pi/2$, we have

$$\begin{aligned} f_t(x) - f_t(y) &= (f(x) - f(y))\,\cos t + (h(x) - h(y))\,\sin t \\ &\leq \omega_f(r)\,\cos t + (h(x) - h(y))\,\sin t. \end{aligned}$$

However, the mean value inequality implies that

$$\begin{aligned} h^2(x) &\leq \frac{\lambda}{V_x(b^{-1}(1-r))} \int_{B_x((1-r))} h^2 \\ &\leq \frac{\lambda}{V_x(b^{-1}(1-r))} \int_{B_p(1)} h^2 \\ &= \frac{\lambda}{V_x(b^{-1}(1-r))}. \end{aligned}$$

Since the same estimate applies to the point y, we conclude that

$$\omega_{f_t}(r) \le \omega_f(r)\, \cos t + C\, \left(\inf_{x\in B_p(r)} V_x(b^{-1}(1-r))\right)^{-\frac{1}{2}}\, \sin t.$$

Letting $r \to 0$, we have

$$\epsilon_{f_t} \le \epsilon_f\, \cos t + C\, \left(V_p(b^{-1})\right)^{-\frac{1}{2}}\, \sin t.$$

Obviously, by reversing the role of f and f_t, we see that $\epsilon_{f_t} \to \epsilon_f$ as $t \to 0$, which proves the continuity of ϵ_f and the existence of ϵ.

The above argument also proved that

$$\begin{aligned}\omega_f\left(\frac{r}{4b}\right) &\le C\, \left(\inf_{x\in B_p\left(\frac{r}{4b}\right)} V_x\left(\frac{3r}{4b}\right)\right)^{-\frac{1}{2}}\, \left(\int_{B_p(r)} f^2\right)^{\frac{1}{2}}\\ &\le C\, V_p^{-\frac{1}{2}}\left(\frac{r}{2b}\right)\, \left(\int_{B_p(r)} f^2\right)^{\frac{1}{2}}.\end{aligned}$$

In particular,

$$\epsilon^2 \le C\, V_p^{-1}\left(\frac{r}{2b}\right)\, \int_{B_p(r)} f^2 \tag{4.1}$$

for all $f \in \mathbb{S}(K)$. However, according to Lemma 3, we have

$$\det_{A_1} A_r \le \left(\frac{k\, r^{\beta}}{\lambda}\right)^k$$

for $\beta = \frac{\ln k - \ln\lambda}{\ln(2b+1)}$. By choosing an A_1 orthonormal basis $\{f_1, \ldots, f_k\}$, which diagonalizes A_r, we have

$$\prod_{i=1}^k \int_{B_p(r)} f_i^2 \le \left(\frac{k\, r^{\beta}}{\lambda}\right)^k.$$

Combining with (4.1), we conclude that

$$\epsilon^{2k} \le C^k\, V_p^{-k}\left(\frac{r}{2b}\right)\, \left(\frac{k\, r^{\beta}}{\lambda}\right)^k.$$

Using the assumption on the lower bound of the volume, we have

$$\epsilon^2 \le C\, k\, r^{\beta-\nu}.$$

This gives a contradiction if we choose

$$\ln k > \nu \ln(2b+1) + \ln \lambda$$

and let $r \to 0$. q.e.d.

5. Polynomial Growth L-harmonic Functions

In this section, we will consider elliptic operator of divergence form on a complete Riemannian manifold M given by

$$Lf = \operatorname{div}(A(x)\nabla f)$$

where $A(x) : T_xM \to T_xM$ is an endomorphism of the tangent space satisfying

$$\langle A(x)V, V\rangle > 0$$

for all nonzero $V \in T_xM$. We also assume that in terms of local coordinates, the coefficients of $A(x) = (a_{ij}(x))$ are measurable functions. Moreover, there exists a constant $\Omega > 0$, such that,

$$0 < \phi(r)\, I \le A(x) \le \Phi(r)\, I < \infty \tag{5.1}$$

for all $x \in B_p(\frac{3r}{2}) \setminus B_p(\frac{r}{2})$, with

$$\frac{\Phi(r)}{\phi(r)} \le \Omega. \tag{5.2}$$

Theorem 10. *Let M be a Riemannian manifold with nonnegative Ricci curvature. Let us define*

$$\mathcal{H}_d(L, M) = \Big\{ f \,|\, Lf = 0 \ \text{ and } \ |f|(x) = O(r^d(x)) \Big\}$$

to be the space of L-harmonic functions with polynomial growth of degree at most d. Under the ellipticity assumption (5.1) and (5.2), the space $\mathcal{H}_d(L, M)$ must be of finite dimension. In fact, there exist constants $C_1(\Omega, n) > 0$ and $C_2 > 0$ depending only on the described variables such that

$$\dim \mathcal{H}_d(L, \mathbb{R}^n) \le C_1 \exp(C_2 d).$$

Proof. The strategy is to observe that if f is an L-harmonic function, then f^2 still satisfies the maximum principle. To use the remark preceding Lemma 3, we need to verify that the weak mean value inequality is valid for the operator L. This is indeed the case if one applies the Moser iteration argument on the ball $B_q(\frac{r}{2})$ with q being distance r from the fixed point $p \in M$. In this case, it is known that only the ratio of the upper and lower bounds of the ellipticity constants, Ω, is involved in the dependency of λ. Since we are using the background Riemannian measure, the ratio

$$\begin{aligned}\frac{V_p(r)}{V_q(\frac{r}{2b})} &\le \frac{V_q(2r)}{V_q(\frac{r}{2b})} \\ &\le (4b)^n\end{aligned}$$

by the volume comparison theorem. Hence the estimate (1.4) becomes

$$\mathrm{tr}_{A_{\frac{3r}{2}}} A_r \le \lambda\,(4b)^n.$$

Applying Lemma 3, rather its argument, and letting $b = \frac{5}{4}$, we obtain for all $r < R$,

$$\det{}_{A_R} A_r \le \left(\frac{k\, r^\beta}{\lambda\, 5^n\, R^\beta}\right)^k$$

if $k = \dim \mathcal{H}_d(L, M)$, with

$$\beta = \frac{\ln k - \ln(\lambda\, 5^n)}{\ln(\frac{3}{2})}. \tag{5.3}$$

However, the growth assumption of $\mathcal{H}_d(L, M)$ implies that

$$\det{}_{A_r} A_R \le C R^{k(2d+n)}.$$

Hence, together with (5.3), we conclude that

$$\beta \le 2d + n,$$

which proves that k must be finite and bounded by

$$k \le C_1\, \exp(C_2\, d)$$

for some constants C_1, $C_2 > 0$. q.e.d.

We would like to point out that in the previous proof, we have only used the fact that M has a Sobolev inequality and satisfies the volume comparison theorem. These two conditions can replace the assumption that M has nonnegative Ricci curvature. In particular, if M is quasi-isometric to a manifold with nonnegative Ricci curvature, then these two properties will hold.

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